

# HARDY-SOBOLEV-MAZ'YA INEQUALITIES FOR ARBITRARY DOMAINS

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**ABSTRACT.** We prove a Hardy-Sobolev-Maz'ya inequality for arbitrary domains  $\Omega \subset \mathbb{R}^N$  with a constant depending only on the dimension  $N \geq 3$ . In particular, for convex domains this settles a conjecture by Filippas, Maz'ya and Tertikas. As an application we derive Hardy-Lieb-Thirring inequalities for eigenvalues of Schrödinger operators on domains.

## 1. INTRODUCTION AND MAIN RESULT

**1.1. Hardy-Sobolev-Maz'ya inequalities.** Hardy inequalities and Sobolev inequalities bound the size of a function, measured by a (possibly weighted)  $L^q$  norm, in terms of its smoothness, measured by an integral of its gradient. Maz'ya [22] proved that for functions on the half-space  $\mathbb{R}_+^N = \{x \in \mathbb{R}^N : x_N > 0\}$ ,  $N \geq 3$ , which vanish on the boundary, the sharp version of the Hardy inequality can be combined with the Sobolev inequality into a single inequality, namely,

$$\int_{\mathbb{R}_+^N} \left( |\nabla u|^2 - \frac{|u|^2}{4x_N^2} \right) dx \geq \sigma_N \left( \int_{\mathbb{R}_+^N} |u|^{\frac{2N}{N-2}} dx \right)^{\frac{N-2}{N}}, \quad u \in C_0^\infty(\mathbb{R}_+^N). \quad (1.1)$$

This inequality, its generalizations to different powers of the gradient [2] and its optimal constants [23, 3] have attracted attention recently. Another series of papers investigates extensions of Hardy's inequality to convex domains and possible  $L^2$ -remainder terms [5, 16, 9, 1]. In [8, 10] Filippas, Maz'ya and Tertikas found an extension of the Hardy-Sobolev-Maz'ya inequality (1.1) to convex domains. They prove that for any convex, bounded domain  $\Omega$  with  $C^2$ -boundary there is a constant  $\sigma(\Omega)$  such that

$$\int_{\Omega} \left( |\nabla u|^2 - \frac{|u|^2}{4 \operatorname{dist}(x, \Omega^c)^2} \right) dx \geq \sigma(\Omega) \left( \int_{\Omega} |u|^{\frac{2N}{N-2}} dx \right)^{\frac{N-2}{N}}, \quad u \in C_0^\infty(\Omega). \quad (1.2)$$

Open problem 1 in [8] asks whether the constant  $\sigma(\Omega)$  can be chosen independently of  $\Omega$ . Our main result is an affirmative answer to this question.

In fact, we shall prove a more general inequality, valid for *any* (not necessarily convex) domain  $\Omega$ . This extension is in the spirit of Davies' paper [6], where non-negativity of the left side of (1.2) was observed. In that paper Davies also introduced

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the weight function

$$D_\Omega(x) := \left( N |\mathbb{S}^{N-1}|^{-1} \int_{\mathbb{S}^{N-1}} d_e(x)^{-2} de \right)^{-\frac{1}{2}},$$

where  $d_e(x) := \inf\{|t| : x + te \in \Omega^c\}$  for  $e \in \mathbb{S}^{N-1}$ , and proves that for any domain  $\Omega \subsetneq \mathbb{R}^N$  one has

$$\int_\Omega |\nabla u|^2 dx \geq \frac{1}{4} \int_\Omega \frac{|u|^2}{D_\Omega^2} dx, \quad u \in C_0^\infty(\Omega). \quad (1.3)$$

The relation between (1.3) and the left side of (1.2) is that

$$D_\Omega(x) \leq \text{dist}(x, \Omega^c) \quad \text{if } \Omega \text{ is convex.} \quad (1.4)$$

This follows by some elementary geometric considerations.

Having introduced all the relevant notation, we are now ready to state our main result.

**Theorem 1.1.** *Let  $N \geq 3$ . There is a constant  $K_N > 0$  such that for any domain  $\Omega \subsetneq \mathbb{R}^N$  and any  $u \in C_0^\infty(\Omega)$*

$$\int_\Omega \left( |\nabla u|^2 - \frac{|u|^2}{4D_\Omega^2} \right) dx \geq K_N \left( \int_\Omega |u|^{\frac{2N}{N-2}} dx \right)^{\frac{N-2}{N}}. \quad (1.5)$$

We emphasize that the constant  $K_N$  does not depend on  $\Omega$ . Hence (1.4) yields (1.2) with a constant independent of  $\Omega$ , thereby solving the problem posed in [8]. Our proof of (1.5) is constructive and gives an explicit value for  $K_N$ . We have nothing to say, however, about its *sharp* value. Is the sharp value of (1.2) given by that in (1.1) for any convex  $\Omega$ ? (This is true if  $\Omega$  is a ball [3].)

If  $\Omega$  has finite measure, then (1.5) implies by means of Hölder's inequality that

$$\int_\Omega \left( |\nabla u|^2 - \frac{|u|^2}{4D_\Omega^2} \right) dx \geq K_N |\Omega|^{-\frac{2}{N}} \int_\Omega |u|^2 dx. \quad (1.6)$$

This inequality for convex  $\Omega$  and with  $D_\Omega(x)$  replaced by  $\text{dist}(x, \Omega^c)$  was the original question posed in the influential paper [5] by Brezis and Marcus. As an answer inequality (1.6) was proved in [16]; see also [9, 1] for further developments.

Another application of Hölder's inequality to (1.5) yields

$$\left( \int_\Omega \left( |\nabla u|^2 - \frac{|u|^2}{4D_\Omega^2} \right) dx \right)^\theta \left( \int_\Omega |u|^2 dx \right)^{1-\theta} \geq K_N^\theta \left( \int_\Omega |u|^q dx \right)^{\frac{2}{q}}, \quad \theta = \frac{N}{2} \left( 1 - \frac{2}{q} \right), \quad (1.7)$$

for all  $2 \leq q \leq \frac{2N}{N-2}$ . It turns out that this is the correct substitute of (1.5) in dimensions one and two.

**Theorem 1.2.** *Let  $N = 1$  or  $N = 2$  and let  $2 \leq q < \infty$ . Then there is a constant  $K_{N,\theta} > 0$  such that for any domain  $\Omega \subsetneq \mathbb{R}^N$  and any  $u \in C_0^\infty(\Omega)$*

$$\left( \int_{\Omega} \left( |\nabla u|^2 - \frac{|u|^2}{4D_{\Omega}^2} \right) dx \right)^{\theta} \left( \int_{\Omega} |u|^2 dx \right)^{1-\theta} \geq K_{N,\theta} \left( \int_{\Omega} |u|^q dx \right)^{\frac{2}{q}}, \quad \theta = \frac{N}{2} \left( 1 - \frac{2}{q} \right). \quad (1.8)$$

Of course, (1.8) implies (1.6) also in dimensions one and two.

We also have a generalization of Theorem 1.1 to powers  $p \geq 2$  of the gradient. The relevant weight function is now

$$D_{\Omega,p}(x) := \left( \frac{\sqrt{\pi} \Gamma(\frac{N+p}{2})}{\Gamma(\frac{p+1}{2}) \Gamma(\frac{N}{2})} |\mathbb{S}^{N-1}|^{-1} \int_{\mathbb{S}^{N-1}} d_e(x)^{-p} de \right)^{-\frac{1}{p}}$$

with  $d_e(x)$  as before. We note that for  $p = 2$  one has  $D_{\Omega,2} = D_{\Omega}$ . The analogue of (1.3), which is valid for any  $p > 1$  and any open domain  $\Omega \subsetneq \mathbb{R}^N$ , is

$$\int_{\Omega} |\nabla u|^p dx \geq \left( \frac{p-1}{p} \right)^p \int_{\Omega} \frac{|u|^p}{(D_{\Omega,p})^p} dx, \quad u \in C_0^\infty(\Omega). \quad (1.9)$$

This implies, in particular, the  $L^p$  Hardy inequality for convex domains [21], since by the same argument leading to (1.4) one sees that

$$D_{\Omega,p}(x) \leq \text{dist}(x, \Omega^c) \quad \text{if } \Omega \text{ is convex.} \quad (1.10)$$

Hardy-Sobolev inequalities for  $p > 2$  and *smooth, convex* domains were also studied in [10]. The following theorem extends this to *arbitrary* domains.

**Theorem 1.3.** *Let  $2 \leq p < N$ . There is a constant  $K_{N,p}$  such that for any domain  $\Omega \subsetneq \mathbb{R}^N$  and any  $u \in C_0^\infty(\Omega)$*

$$\int_{\Omega} \left( |\nabla u|^p - \left( \frac{p-1}{p} \right)^p \frac{|u|^p}{(D_{\Omega,p})^p} \right) dx \geq K_{N,p} \left( \int_{\Omega} |u|^{\frac{Np}{N-p}} dx \right)^{\frac{N-p}{N}}. \quad (1.11)$$

Inequalities analogous to (1.6) and (1.8) hold for  $p > 2$  as well.

We shall give the details of the proofs of Theorems 1.1, 1.2 and 1.3 in Section 2. Before summarizing the proof strategy in Subsection 1.3, we would first like to give an application of these theorems to spectral problems in mathematical physics.

**1.2. Hardy-Lieb-Thirring inequalities.** To motivate the following inequalities we introduce a ‘duality parameter’ (physically: a potential)  $V \in L^{N/2}(\Omega)$ . If  $N \geq 3$  we can infer from Hölder’s inequality and (1.5) that

$$\begin{aligned} \int_{\Omega} \left( |\nabla u|^2 - \frac{|u|^2}{4D_{\Omega}^2} + V|u|^2 \right) dx &\geq \int_{\Omega} \left( |\nabla u|^2 - \frac{|u|^2}{4D_{\Omega}^2} \right) dx - \|V_-\|_{\frac{N}{2}} \|u\|_{\frac{2N}{N-2}}^2 \\ &\geq \int_{\Omega} \left( |\nabla u|^2 - \frac{|u|^2}{4D_{\Omega}^2} \right) dx \left( 1 - K_N^{-1} \|V_-\|_{\frac{N}{2}} \right), \end{aligned}$$

where  $V_- = \max\{-V, 0\}$  denotes the negative part of the potential. From this we conclude that the Schrödinger operator

$$-\Delta - (2D_\Omega)^{-2} + V \quad \text{in } L^2(\Omega) \quad (1.12)$$

with Dirichlet boundary conditions on  $\partial\Omega$  has no negative eigenvalues if the potential satisfies  $\|V_-\|_{\frac{N}{2}} \leq K_N$ . The following theorem improves this by saying that not only the existence of negative eigenvalues but even their *total number* is controlled in terms of the  $L^{N/2}$ -norm of the potential. In the case of the ‘usual’ Schrödinger operator  $-\Delta + V$ , this is the famous inequality of Cwikel, Lieb and Rozenblum. We refer to the reviews [18, 17] for references, motivations and applications of this inequality. Our new result is that this inequality remains valid (possibly up to a constant), even when the positive term  $(2D_\Omega)^{-2}$  is subtracted from the Laplacian. The precise statement is

**Theorem 1.4.** *Let  $N \geq 3$ . There is a constant  $L_N$  such that for any domain  $\Omega \subsetneq \mathbb{R}^N$  and any  $V \in L^{N/2}(\Omega)$  the number  $N(-\Delta - (2D_\Omega)^{-2} + V)$  of negative eigenvalues (including multiplicities) of the operator (1.12) is bounded by*

$$N(-\Delta - (2D_\Omega)^{-2} + V) \leq L_N \int_\Omega V_-^{\frac{N}{2}} dx. \quad (1.13)$$

*This inequality holds with  $L_N = e^{\frac{N}{2}-1} K_N^{-\frac{N}{2}}$ , where  $K_N$  is the constant from (1.5).*

For example, choosing  $V = (2D_\Omega)^{-2} - \mu$ , where  $\mu$  is a positive constant, we infer that the number of eigenvalues less than  $\mu$  of the Dirichlet Laplacian on  $\Omega$  is bounded by

$$N(-\Delta - \mu) \leq L_N \int_\Omega ((2D_\Omega)^{-2} - \mu)_-^{\frac{N}{2}} dx.$$

Since the latter integral can be bounded as follows,

$$\int_\Omega ((2D_\Omega)^{-2} - \mu)_-^{\frac{N}{2}} dx \leq \mu^{\frac{N}{2}} \left| \{x \in \Omega : D_\Omega(x) > (4\mu)^{-\frac{1}{2}}\} \right|,$$

this quantifies in a uniform way the observation that because of the Dirichlet conditions most of the eigenvalues come from the bulk of  $\Omega$ .

It is well-known that an inequality of the form (1.13) implies inequalities for moments of the negative eigenvalues  $E_j$  of the operator (1.12), namely,

$$\sum_j |E_j|^\gamma \leq L_{N,\gamma} \int_\Omega V_-^{\gamma + \frac{N}{2}} dx. \quad (1.14)$$

Here  $\gamma > 0$ , and the sum runs over all (including multiplicities) negative eigenvalues of  $-\Delta - (2D_\Omega)^{-2} + V$ . When the term  $(2D_\Omega)^{-2}$  is absent, these inequalities go back to Lieb and Thirring [20]. Just like (1.7) is the appropriate consequence of (1.5) that can be generalized to dimensions one and two, inequality (1.14) remains valid in these dimensions.

**Theorem 1.5.** *Let  $\gamma > 1/2$  if  $N = 1$  and  $\gamma > 0$  if  $N = 2$ . Then there is a constant  $L_{N,\gamma}$  such that for any domain  $\Omega \subsetneq \mathbb{R}^N$  and any  $V \in L^{\gamma+N/2}(\Omega)$  the negative eigenvalues  $E_j$  of the operator (1.12) are bounded by (1.14).*

It is quite likely that (1.14) remains valid for  $N = 1$  and  $\gamma = 1/2$ , but we did not try to prove this. We emphasize that the main point of (1.14) is its universality, being valid for any  $\Omega$  and  $V$  and even for small values of  $\gamma$ . On the other hand, in the special case of  $V \equiv \text{const}$  and for  $\gamma \geq 3/2$ , much more precise information about the influence of the boundary on Lieb-Thirring inequalities is available; see, e.g., the recent paper [15] and references therein.

Hardy-Lieb-Thirring inequalities of the form (1.14), but with a Hardy term becoming singular at a single point, were first derived in [7] and found later an application to the physical problem of stability of matter [12]; see also [11]. The papers [12] and [13] develop an approach how to deduce Lieb-Thirring-type inequalities from (a-priori weaker) Sobolev-type inequalities. Theorems 1.4 and 1.5, which appear here for the first time, were actually a main motivation for developing this abstract approach.

**1.3. Strategy of the proof.** In order to motivate our argument, we first review the classical proof by Gagliardo and Nirenberg of the Sobolev inequality. For simplicity we restrict ourselves to dimension  $N = 3$  and we want to prove that

$$\int_{\mathbb{R}^3} |\nabla u|^2 dx \geq S \left( \int_{\mathbb{R}^3} |u|^6 dx \right)^{1/3}. \quad (1.15)$$

The starting point of the proof is the one-dimensional Sobolev inequality

$$|f(t)| \leq \frac{1}{2} \int_{\mathbb{R}} |f'| ds, \quad f \in C_0^\infty(\mathbb{R}), \quad (1.16)$$

which comes from the formula  $f(t) = \frac{1}{2} \left( \int_{-\infty}^t f(s) ds - \int_{-\infty}^t f(s) ds \right)$ . Now given a function  $v \in C_0^\infty(\mathbb{R}^3)$  we apply the one-dimensional inequality to the three one-dimensional functions  $t \mapsto v(t, x_2, x_3)$ ,  $t \mapsto v(x_1, t, x_3)$  and  $t \mapsto v(x_1, x_2, t)$  and we obtain

$$|v(x)|^3 \leq \frac{1}{8} \rho_1(x_2, x_3) \rho_2(x_1, x_3) \rho_3(x_1, x_2),$$

where

$$\rho_1(x_2, x_3) := \int_{\mathbb{R}} \left| \frac{\partial}{\partial x_1} v(t, x_2, x_3) \right| dt,$$

and similarly for  $\rho_2$  and  $\rho_3$ . Then the Schwarz and the arithmetic-geometric mean inequalities imply that

$$\int_{\mathbb{R}^3} |v|^{\frac{3}{2}} dx \leq 8^{-\frac{1}{2}} \prod_{j=1}^3 \|\rho_j\|_1^{\frac{1}{2}} \leq 8^{-\frac{1}{2}} \left( \frac{1}{3} \sum_{j=1}^3 \|\rho_j\|_1 \right)^{\frac{3}{2}} = 8^{-\frac{1}{2}} \left( \frac{1}{3} \sum_{j=1}^3 \int_{\mathbb{R}^3} \left| \frac{\partial v}{\partial x_j} \right| dx \right)^{\frac{3}{2}}.$$

This is an  $L^1$  Sobolev inequality, and in order to arrive at the  $L^2$  Sobolev inequality (1.15) we set  $v = u^4$  and estimate

$$\sum_{j=1}^3 \int_{\mathbb{R}^3} \left| \frac{\partial v}{\partial x_j} \right| dx = 4 \sum_{j=1}^3 \int_{\mathbb{R}^3} \left| \frac{\partial u}{\partial x_j} \right| |u|^3 dx \leq 4\sqrt{3} \left( \int_{\mathbb{R}^3} |\nabla u|^2 dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^3} |u|^6 dx \right)^{\frac{1}{2}}.$$

Finally dividing by  $\|u\|_6^3$ , we obtain (1.15).

The simple observation, which is behind our proof of Theorem 1.1, is that one can reverse the order of the steps in the above argument. Namely, one can set already  $f = g^4$  in the one-dimensional Sobolev inequality, which then becomes

$$|g(t)|^4 \leq 2 \left( \int_{\mathbb{R}} |g'|^2 ds \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}} |g|^6 dx \right)^{\frac{1}{2}}, \quad g \in C_0^\infty(\mathbb{R}). \quad (1.17)$$

Now we obtain

$$|u(x)|^{12} \leq 8 \phi_1(x_2, x_3)^{\frac{1}{2}} \psi_1(x_2, x_3)^{\frac{1}{2}} \phi_2(x_1, x_3)^{\frac{1}{2}} \psi_2(x_1, x_3)^{\frac{1}{2}} \phi_3(x_1, x_2)^{\frac{1}{2}} \psi_3(x_1, x_2)^{\frac{1}{2}},$$

with

$$\phi_1(x_2, x_3) := \int_{\mathbb{R}} \left| \frac{\partial}{\partial x_1} u(t, x_2, x_3) \right|^2 dt, \quad \psi_1(x_2, x_3) := \int_{\mathbb{R}} |u(t, x_2, x_3)|^6 dt,$$

and similarly for the remaining functions. As before, from Hölder's inequality we get

$$\int_{\mathbb{R}^3} |u|^6 dx \leq 8^{\frac{1}{2}} \prod_{j=1}^3 \left\| \phi_j^{\frac{1}{2}} \psi_j^{\frac{1}{2}} \right\|_1^{\frac{1}{2}}.$$

Next, we apply the Schwarz inequality  $\|\phi_j^{\frac{1}{2}} \psi_j^{\frac{1}{2}}\|_1 \leq \|\phi_j\|_1^{\frac{1}{2}} \|\psi_j\|_1^{\frac{1}{2}}$ , we note that  $\|\psi_j\|_1 = \|u\|_6^6$  and we apply the geometric-arithmetic mean inequality to conclude that

$$\begin{aligned} \int_{\mathbb{R}^3} |u|^6 dx &\leq 8^{\frac{1}{2}} \|u\|_6^{\frac{9}{2}} \prod_{j=1}^3 \|\phi_j\|_1^{\frac{1}{4}} \leq 8^{\frac{1}{2}} \|u\|_6^{\frac{9}{2}} \left( \frac{1}{3} \sum_{j=1}^3 \|\phi_j\|_1 \right)^{\frac{3}{4}} \\ &= 8^{\frac{1}{2}} \|u\|_6^{\frac{9}{2}} \left( \frac{1}{3} \int_{\mathbb{R}^3} |\nabla u|^2 dx \right)^{\frac{3}{4}}, \end{aligned}$$

which is our desired goal (1.15).

The upshot of this discussion is that in order to arrive at the  $L^2$  Sobolev inequality (1.15) (which is weaker than the  $L^1$  Sobolev inequality) we only need the one-dimensional  $L^2$  Sobolev inequality (1.17), and not the one-dimensional  $L^1$  Sobolev inequality (1.16). This simple observation is of relevance for us because there is not even a Hardy inequality, i.e., the inequality

$$\int_{-1}^1 |f'(x)| dx \geq C \int_{-1}^1 \frac{|f(x)|}{1-|x|} dx$$

is false no matter how small the constant  $C$ . However, and this is our technical key result (Proposition 2.1), we can prove a one-dimensional  $L^2$  Sobolev-type inequality with a Hardy term! In fact such inequalities hold for all  $p \geq 2$  (Proposition 2.5). For

$1 < p < 2$  there is a Hardy inequality, however, it is not known whether any version of Proposition 2.5 might hold for  $1 < p < 2$ . Once such an inequality is established, the rest of the argument outlined above yields Hardy-Sobolev-Mazy's inequalities also for  $1 < p < 2$ .

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## 2. PROOF OF THEOREMS 1.1, 1.2 AND 1.3

**2.1. A one-dimensional inequality.** The following inequality is the key for proving Theorem 1.1.

**Proposition 2.1.** *Let  $q \geq 2$ . There is a constant  $C_q$  such that for every  $f \in C_0^\infty(-1, 1)$  and for every  $t \in [-1, 1]$  one has*

$$|f(t)|^{q+2} \leq C_q \int_{-1}^1 \left( |f'|^2 - \frac{|f|^2}{4(1-|s|)^2} \right) ds \int_{-1}^1 |f|^q ds, \quad (2.1)$$

with  $C_q \leq (q+2)^2$ .

*Proof.* We begin by noting that if we write  $f(t) = \sqrt{1-|t|} g(t)$ , then what we have to show is that

$$|g(t)|^{q+2} \leq (q+2)^2 (1-|t|)^{-\frac{q+2}{2}} \left( \int_{-1}^1 |g'|^2 (1-|s|) ds + |g(0)|^2 \right) \int_{-1}^1 |g|^q (1-|s|)^{\frac{q}{2}} ds.$$

By symmetry it suffices to prove this for  $t \in [0, 1]$  only. Now for such  $t$  we can use the fact that  $(1-t)^{\frac{q+2}{4}}$  is decreasing and we find that

$$\begin{aligned} |g(t)|^{\frac{q+2}{2}} - |g(0)|^{\frac{q+2}{2}} &\leq \frac{q+2}{2} \int_0^t |g|^{\frac{q}{2}} |g'| ds \\ &\leq \frac{q+2}{2} (1-t)^{-\frac{q+2}{4}} \int_0^1 |g|^{\frac{q}{2}} |g'| (1-s)^{\frac{q+2}{4}} ds \\ &\leq \frac{q+2}{2} (1-t)^{-\frac{q+2}{4}} \left( \int_0^1 |g'|^2 (1-s) ds \right)^{\frac{1}{2}} \left( \int_0^1 |g|^q (1-s)^{\frac{q}{2}} ds \right)^{\frac{1}{2}}. \end{aligned}$$

Thus it remains to show that

$$|g(0)|^{q+2} \leq \frac{(q+2)^2}{4} \left( \int_{-1}^1 |g'|^2 (1-|s|) ds + |g(0)|^2 \right) \int_{-1}^1 |g|^q (1-|s|)^{\frac{q}{2}} ds. \quad (2.2)$$

Of course, it suffices to show this if  $g$  is non-negative, which is what we will assume henceforth. Let  $\alpha$  be a parameter (to be specified later). Since  $(1-s)^\alpha g(s)^{(q+2)/2}$  vanishes near  $s = 1$  we can write

$$\begin{aligned} |g(0)|^{\frac{q+2}{2}} &= - \int_0^1 \left( \frac{q+2}{2} (1-s)^\alpha g(s)^{\frac{q}{2}} g'(s) - \alpha (1-s)^{\alpha-1} g(s)^{\frac{q+2}{2}} \right) ds \\ &= - \int_0^1 g(s)^{\frac{q}{2}} (1-s)^{\frac{q}{4}} \left( \frac{q+2}{2} (1-s)^{\alpha-\frac{q}{4}} g'(s) - \alpha g(s) (1-s)^{\alpha-\frac{q+4}{4}} \right) ds. \end{aligned}$$

Using the Schwarz inequality we find

$$|g(0)|^{\frac{q+2}{2}} \leq \left( \int_0^1 g(s)^q (1-s)^{\frac{q}{2}} ds \right)^{1/2} T^{1/2}$$

with

$$\begin{aligned} T &= \int_0^1 \left( \frac{(q+2)^2}{4} (1-s)^{2\alpha-\frac{q}{2}} g'^2 - (q+2)\alpha(1-s)^{2\alpha-\frac{q+2}{2}} g g' + \alpha^2 g^2 (1-s)^{2\alpha-\frac{q+4}{2}} \right) ds \\ &= \int_0^1 \left( \frac{(q+2)^2}{4} (1-s)^{2\alpha-\frac{q}{2}} g'^2 - \frac{q+2}{2} \alpha (1-s)^{2\alpha-\frac{q+2}{2}} (g^2)' + \alpha^2 g^2 (1-s)^{2\alpha-\frac{q+4}{2}} \right) ds \\ &= \int_0^1 \left( \frac{(q+2)^2}{4} (1-s)^{2\alpha-\frac{q}{2}} g'^2 + \alpha \left( \alpha - \frac{q+2}{2} (2\alpha - \frac{q+2}{2}) \right) (1-s)^{2\alpha-\frac{q+4}{2}} g^2 \right) ds + \alpha \frac{q+2}{2} g(0)^2. \end{aligned}$$

Now we pick  $\alpha$  such that  $\alpha - \frac{q+2}{2} (2\alpha - \frac{q+2}{2}) = 0$ , which leads to

$$\alpha = \frac{(q+2)^2}{4(q+1)}.$$

Hence we have

$$|g(0)|^{\frac{q+2}{2}} \leq \frac{q+2}{2} \left( \int_0^1 g^q (1-s)^{\frac{q}{2}} ds \right)^{1/2} \left( \int_0^1 g'(s)^2 (1-s)^{\frac{(q+2)^2}{2(q+1)} - \frac{q}{2}} ds + \frac{q+2}{2(q+1)} g(0)^2 \right)^{1/2},$$

which, since  $\frac{(q+2)^2}{2(q+1)} - \frac{q}{2} \geq 1$  and  $\frac{q+2}{2(q+1)} \leq 1$ , is bounded above by

$$\frac{q+2}{2} \left( \int_0^1 (1-s)^{\frac{q}{2}} g(s)^q ds \right)^{1/2} \left( \int_0^1 g'(s)^2 (1-s) ds + g(0)^2 \right)^{1/2}.$$

This proves the claimed inequality (2.2).  $\square$

**Corollary 2.2.** *Let  $q \geq 2$ . Then, with the same constant  $C_q$  as in (2.1), one has for every open set  $\Omega \subsetneq \mathbb{R}$  and for every  $f \in C_0^\infty(\Omega)$*

$$\sup_{t \in \Omega} |f(t)|^{q+2} \leq C_q \int_{\Omega} \left( |f'|^2 - \frac{|f|^2}{4 \operatorname{dist}(t, \Omega^c)^2} \right) dt \int_{\Omega} |f|^q dt. \quad (2.3)$$

*Proof.* First, if  $\Omega$  is an interval, then (2.3) follows from (2.1) by a translation and a dilation. Now the extension to arbitrary open sets (that is, countable unions of disjoint intervals) is straightforward.  $\square$

The following inequality will be needed to deal with the two dimensional case.

**Corollary 2.3.** *Let  $q \geq 4$ . Then, with the same constant  $C_q$  as in (2.1), one has for every open set  $\Omega \subsetneq \mathbb{R}$  and for every  $f \in C_0^\infty(\Omega)$*

$$\sup_{t \in \Omega} |f(t)|^q \leq C_{q-2} \int_{\Omega} \left( |f'|^2 - \frac{|f|^2}{4 \operatorname{dist}(t, \Omega^c)^2} \right) dt \left( \int_{\Omega} |f|^2 dt \right)^{\frac{2}{q-2}} \left( \int_{\Omega} |f|^q dt \right)^{\frac{q-4}{q-2}}.$$

*Proof.* We apply Corollary 2.2 with  $q$  replaced by  $q-2$  and we estimate  $\|f\|_{q-2}^{q-2}$  using Hölder's inequality.  $\square$



**2.2. The inequality in dimensions  $N \geq 3$ .** In order to pass from the one-dimensional inequality of Corollary 2.2 to Theorem 1.1 we use the well-known argument of Gagliardo and Nirenberg. We shall use the following notation for  $x \in \mathbb{R}^N$  and  $1 \leq j \leq N$ ,

$$\tilde{x}_j = (x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_N) \in \mathbb{R}^{N-1}.$$

Then one has

**Lemma 2.4.** *Let  $N \geq 2$  and let  $f_1, \dots, f_N \in L^{N-1}(\mathbb{R}^N)$ . Then the function  $f(x) := f_1(\tilde{x}_1) \cdots f_N(\tilde{x}_N)$  belongs to  $L^1(\mathbb{R}^N)$  and*

$$\|f\|_{L^1(\mathbb{R}^N)} \leq \prod_{j=1}^N \|f_j\|_{L^{N-1}(\mathbb{R}^N)}.$$

The easy proof, based on Hölder's inequality, can be found for instance in [4].

With Lemma 2.4 at hand we now are ready to give the *proof of Theorem 1.1*. Let  $e_1, \dots, e_N$  be the standard unit vectors in  $\mathbb{R}^N$ . For a given domain  $\Omega \subsetneq \mathbb{R}^N$  we write  $d_j$  instead of  $d_{e_j}$ , that is,

$$d_j(x) = \inf\{|t| : x + te_j \in \Omega^c\}.$$

Now if  $u \in C_0^\infty(\Omega)$ , then Corollary (2.3) yields

$$|u(x)| \leq C_q (g_j(\tilde{x}_j) h_j(\tilde{x}_j))^{\frac{N-2}{4(N-1)}}$$

for any  $1 \leq j \leq N$ , where

$$g_j(\tilde{x}_j) := \int_{\mathbb{R}} \left( \left| \frac{\partial u}{\partial x_j}(x) \right|^2 - \frac{|u(x)|^2}{4d_j(x)^2} \right) dx_j \quad \text{and} \quad h_j(\tilde{x}_j) := \int_{\mathbb{R}} |u(x)|^q dx_j$$

with  $q = \frac{2N}{N-2}$ . Thus

$$|u(x)|^N \leq C_q^N \prod_{j=1}^N (g_j(\tilde{x}_j) h_j(\tilde{x}_j))^{\frac{N-2}{4(N-1)}},$$

or, what is the same,

$$|u(x)|^q \leq C_q^q \prod_{j=1}^N (g_j(\tilde{x}_j) h_j(\tilde{x}_j))^{\frac{1}{2(N-1)}}.$$

From Lemma 2.4 we infer that

$$\int_{\mathbb{R}^N} |u(x)|^q dx \leq C_q^q \prod_{j=1}^N \left( \int_{\mathbb{R}^{N-1}} \sqrt{g_j(y) h_j(y)} dy \right)^{\frac{1}{N-1}}.$$

Now we use the fact that

$$\|h_j\|_{L^1(\mathbb{R}^{N-1})} = \|u\|_{L^q(\mathbb{R}^N)}^q \quad \text{for every } j = 1, \dots, N,$$

and derive from the Schwarz and the arithmetic-geometric mean inequality that

$$\begin{aligned} \prod_{j=1}^N \int_{\mathbb{R}^{N-1}} \sqrt{g_j(y) h_j(y)} dy &\leq \prod_{j=1}^N \|g_j\|_1^{\frac{1}{2}} \|h_j\|_1^{\frac{1}{2}} = \|u\|_q^{\frac{Nq}{2}} \prod_{j=1}^N \|g_j\|_1^{\frac{1}{2}} \\ &\leq \|u\|_q^{\frac{Nq}{2}} \left( N^{-1} \sum_{j=1}^N \|g_j\|_1 \right)^{\frac{N}{2}}. \end{aligned}$$

To summarize, we have shown that

$$\int_{\mathbb{R}^N} |u(x)|^q dx \leq C_q^q \|u\|_q^{\frac{Nq}{2(N-1)}} \left( N^{-1} \sum_{j=1}^N \|g_j\|_1 \right)^{\frac{N}{2(N-1)}},$$

that is,

$$\begin{aligned} \left( \int_{\mathbb{R}^N} |u(x)|^q dx \right)^{\frac{2}{q}} &\leq C_q^{\frac{4(N-1)}{N-2}} N^{-1} \sum_{j=1}^N \|g_j\|_1 \\ &= C_q^{\frac{4(N-1)}{N-2}} N^{-1} \int_{\Omega} \left( |\nabla u|^2 - \frac{1}{4} \sum_{j=1}^N \frac{|u|^2}{d_j^2} \right) dx. \end{aligned}$$

Finally, as in [6], we average over all choices of the coordinate system and obtain the inequality claimed in Theorem 1.1.  $\square$

**2.3. The inequality in dimensions one and two.** Next, we prove Theorem 1.2.

*The case  $N = 1$ .* We bound  $\|f\|_q^q \leq \|f\|_{\infty}^{q-2} \|f\|_2^2$  and apply Corollary 2.2 to obtain

$$\int |f|^q dt \leq \|f\|_{\infty}^{q-2} \|f\|_2^2 \leq C_q^{\frac{q-2}{q+2}} \left( \int_{\Omega} \left( |f'|^2 - \frac{|f|^2}{4 \operatorname{dist}(t, \Omega^c)^2} \right) dt \right)^{\frac{q-2}{q+2}} \|f\|_q^{\frac{q(q-2)}{q+2}} \|f\|_2^2.$$

This is the inequality claimed in Theorem 1.2.

*The case  $N = 2$ .* Here we proceed similarly to the case  $N \geq 3$ . We first observe that by Hölder's inequality it suffices to prove the inequality only for large  $q$ , say  $q \geq 4$ . For such  $q$  we can apply Corollary 2.3 and obtain

$$|u(x)|^q \leq C_{q-2}^q \prod_{j=1}^2 \left( g_j(\tilde{x}_j)^{\frac{1}{2}} h_j(\tilde{x}_j)^{\frac{q-4}{2(q-2)}} k_j(\tilde{x}_j)^{\frac{1}{q-2}} \right),$$

where  $g_j$  and  $h_j$  are defined as before and where

$$k_j(\tilde{x}_j) := \int_{\mathbb{R}} |u(x)|^2 dx_j.$$

We integrate this inequality over  $\mathbb{R}^2$  (note that Lemma 2.4 is trivial for  $N = 2$ ) and bound using Hölder's inequality,

$$\begin{aligned} \int_{\mathbb{R}^2} |u(x)|^q dx &\leq C_{q-2}^q \prod_{j=1}^2 \int_{\mathbb{R}} g_j(y)^{\frac{1}{2}} h_j(y)^{\frac{q-4}{2(q-2)}} k_j(y)^{\frac{1}{q-2}} dy \\ &\leq C_{q-2}^q \prod_{j=1}^2 \|g_j\|_1^{\frac{1}{2}} \|h_j\|_1^{\frac{q-4}{2(q-2)}} \|k_j\|_1^{\frac{1}{q-2}} \\ &= C_{q-2}^q \|u\|_q^{\frac{q(q-4)}{q-2}} \|u\|_2^{\frac{4}{q-2}} \prod_{j=1}^2 \|g_j\|_1^{\frac{1}{2}}. \end{aligned}$$

The claimed inequality now follows as before by the arithmetic-geometric mean inequality for  $\prod_j \|g_j\|_1$  and by averaging over all coordinate systems.  $\square$

**2.4. The case  $p > 2$ .** The analogue of Proposition 2.1 is

**Proposition 2.5.** *Let  $q \geq p \geq 2$ . There is a constant  $C_{p,q}$  such that for every  $f \in C_0^\infty(-1, 1)$  and for every  $t \in [-1, 1]$  one has*

$$|f(t)|^{q(p-1)+p} \leq C_{p,q} \int_{-1}^1 \left( |f'|^p - \left( \frac{p-1}{p} \right)^p \frac{|f|^p}{(1-|s|)^p} \right) ds \left( \int_{-1}^1 |f|^q ds \right)^{p-1}. \quad (2.4)$$

Given this inequality, Theorem 1.3 follows again by the Gagliardo-Nirenberg argument as in the previous subsections, but now with  $q = Np/(N-p)$ . We omit the details, we only point out that the constant in the definition of  $D_{\Omega,p}$  appears through the evaluation of the integral

$$|\mathbb{S}^{N-1}|^{-1} \int_{\mathbb{S}^{N-1}} |a \cdot e|^p de = \frac{\Gamma(\frac{p+1}{2}) \Gamma(\frac{N}{2})}{\sqrt{\pi} \Gamma(\frac{N+p}{2})} |a|^p$$

for  $a \in \mathbb{R}^N$ .

*Proof.* We shall use that  $|a+b|^p \geq |a|^p + p|a|^{p-2} \operatorname{Re} \bar{a}b + c_p|b|^p$  for all  $a, b \in \mathbb{C}$  and some explicit  $c_p$ , see [14]. (Here we use that  $p \geq 2$ .) Hence if we write  $f(t) = (1-|t|)^{(p-1)/p} g(t)$ , then

$$\begin{aligned} &\int_{-1}^1 \left( |f'|^p - \left( \frac{p-1}{p} \right)^p \frac{|f|^p}{(1-|s|)^p} \right) ds \\ &= \int_{-1}^1 \left( \left| (1-|s|)^{(p-1)/p} g' - \frac{p-1}{p} (\operatorname{sgn} s) (1-|s|)^{-1/p} g \right|^p - \left( \frac{p-1}{p} \right)^p \frac{|g|^p}{1-|s|} \right) ds \\ &\geq \int_{-1}^1 \left( -p \left( \frac{p-1}{p} \right)^{p-1} (\operatorname{sgn} s) |g|^{p-2} \operatorname{Re} \bar{g} g' + c_p (1-|s|)^{p-1} |g'|^p \right) ds \\ &= 2 \left( \frac{p-1}{p} \right)^{p-1} |g(0)|^p + c_p \int_{-1}^1 (1-|s|)^{p-1} |g'|^p ds. \end{aligned}$$

Thus it is enough to show that

$$|g(t)|^{q(p-1)+p} \leq C(1-|t|)^{-\frac{(p-1)(p+q(p-1))}{p}} \left( \int_{-1}^1 (1-|s|)^{p-1} |g'|^p ds + d|g(0)|^p \right) \\ \times \left( \int_{-1}^1 |g|^q (1-|s|)^{\frac{q(p-1)}{p}} ds \right)^{p-1}.$$

where  $d = 2c_p^{-1}(\frac{p-1}{p})^{p-1}$ . By symmetry it suffices to prove this for  $t \in [0, 1]$  only. Now for such  $t$  we can use the fact that  $(1-t)^{\frac{(p-1)(q(p-1)+p)}{p^2}}$  is decreasing and we find that

$$|g(t)|^{\frac{q(p-1)+p}{p}} - |g(0)|^{\frac{q(p-1)+p}{p}} \leq \frac{q(p-1)+p}{p} \int_0^t |g|^{\frac{q(p-1)}{p}} |g'| ds \\ \leq \frac{q(p-1)+p}{p} (1-t)^{-\frac{(p-1)(p+q(p-1))}{p^2}} \int_0^1 |g|^{\frac{q(p-1)}{p}} |g'| (1-s)^{\frac{(p-1)(p+q(p-1))}{p^2}} ds \\ \leq \frac{q(p-1)+p}{p} (1-t)^{-\frac{(p-1)(p+q(p-1))}{p^2}} \left( \int_0^1 |g'|^p (1-s)^{p-1} ds \right)^{\frac{1}{p}} \left( \int_0^1 |g|^q (1-s)^{\frac{q(p-1)}{p}} ds \right)^{\frac{p-1}{p}}.$$

Thus it remains to show that

$$|g(0)|^{q(p-1)+p} \leq C \left( \int_{-1}^1 |g'|^p (1-|t|)^{p-1} dt + d|g(0)|^p \right) \left( \int_{-1}^1 |g|^q (1-|t|)^{\frac{q(p-1)}{p}} dt \right)^{p-1}. \quad (2.5)$$

In order to prove this, we choose a free parameter  $T \in (0, 1)$  and a Lipschitz function  $\chi$  with  $0 \leq \chi \leq 1$ ,  $\chi(0) = 1$  and  $\chi(t) = 0$  for  $|t| \in [T, 1]$ . We put

$$L := \left( \int_{-1}^1 |\chi'|^{\frac{pq}{q-p}} ds \right)^{\frac{pq}{q-p}}.$$

Now we choose another parameter  $A$  (which will be fixed later depending on  $T$  and  $L$ ) and distinguish two cases according to whether

$$|g(0)|^q \leq A^{\frac{p}{p-1}} \int_{-1}^1 |g|^q (1-|s|)^{\frac{q(p-1)}{p}} ds \quad (2.6)$$

or not. In the first case, we can trivially estimate

$$|g(0)|^{\frac{q(p-1)+p}{p}} \leq A |g(0)| \left( \int_{-1}^1 |g|^q (1-|t|)^{\frac{q(p-1)}{p}} dt \right)^{\frac{p-1}{p}} \\ \leq A d^{-\frac{1}{p}} \left( \int_{-1}^1 |g'|^p (1-|t|)^{p-1} dt + d|g(0)|^p \right)^{\frac{1}{p}} \left( \int_{-1}^1 |g|^q (1-|t|)^{\frac{q(p-1)}{p}} dt \right)^{\frac{p-1}{p}}$$

and we have arrived at our goal (2.5). Now assume that the opposite inequality in (2.6) holds. We define  $g_0 := \chi g$  and estimate this function similarly as above. Indeed,

since  $g_0(T) = g_0(-T) = 0$ ,

$$\begin{aligned}
|g_0(0)|^{\frac{q(p-1)+p}{p}} &\leq \frac{q(p-1)+p}{2p} \int_{-T}^T |g_0|^{\frac{q(p-1)}{p}} |g'_0| ds \\
&\leq \frac{q(p-1)+p}{2p} (1-T)^{-\frac{(p-1)(p+q(p-1))}{p^2}} \int_{-1}^1 |g_0|^{\frac{q(p-1)}{p}} |g'_0| (1-|s|)^{\frac{(p-1)(p+q(p-1))}{p^2}} ds \\
&\leq \frac{q(p-1)+p}{2p} (1-T)^{-\frac{(p-1)(p+q(p-1))}{p^2}} \left( \int_{-1}^1 |g'_0|^p (1-|s|)^{p-1} ds \right)^{\frac{1}{p}} \\
&\quad \times \left( \int_{-1}^1 |g_0|^q (1-|s|)^{\frac{q(p-1)}{p}} ds \right)^{\frac{p-1}{p}}.
\end{aligned}$$

In order to again arrive at (2.5) we recall that  $g_0(0) = g(0)$  and that one has

$$\int_{-1}^1 |g_0|^q (1-|s|)^{\frac{q(p-1)}{p}} ds \leq \int_{-1}^1 |g|^q (1-|s|)^{\frac{q(p-1)}{p}} ds.$$

Finally, we estimate the term involving  $g'_0$  by means of the triangle inequality

$$\begin{aligned}
\left( \int_{-1}^1 |g'_0|^p (1-|s|)^{p-1} ds \right)^{\frac{1}{p}} &\leq \left( \int_{-1}^1 |g'|^p \chi^p (1-|s|)^{p-1} ds \right)^{\frac{1}{p}} \\
&\quad + \left( \int_{-1}^1 |g|^p |\chi'|^p (1-|s|)^{p-1} ds \right)^{\frac{1}{p}} \\
&\leq \left( \int_{-1}^1 |g'|^p (1-|s|)^{p-1} ds \right)^{\frac{1}{p}} \\
&\quad + L \left( \int_{-1}^1 |g|^q (1-|s|)^{\frac{q(p-1)}{p}} ds \right)^{\frac{1}{q}} \\
&\leq \left( \int_{-1}^1 |g'|^p (1-|s|)^{p-1} ds \right)^{\frac{1}{p}} + LA^{-\frac{p}{q(p-1)}} |g(0)| \\
&\leq 2^{\frac{p-1}{p}} \left( \int_{-1}^1 |g'|^p (1-|s|)^{p-1} ds + L^p A^{-\frac{p^2}{q(p-1)}} |g(0)|^p \right)^{\frac{1}{p}},
\end{aligned}$$

where in the next to last step we used the inequality opposite to (2.6). Thus choosing  $A$  large enough so that  $L^p A^{-\frac{p^2}{q(p-1)}} \leq d$  we arrive again at (2.5).  $\square$

### 3. PROOFS OF THEOREMS 1.4 AND 1.5

**3.1. Equivalence of Sobolev and Lieb-Thirring inequalities.** We shall deduce Theorem 1.4 from Theorem 1.1 by applying the abstract approach developed in [13]. Let us briefly summarize the main result of [13]. Let  $X$  be a sigma-finite measure space and let  $t$  be a closed, non-negative quadratic form in  $L^2(X)$  with domain  $\text{dom } t$ . We assume the following

**Assumption 3.1** (Generalized Beurling-Deny conditions).

- (a) if  $u, v \in \text{dom } t$  are real-valued, then  $t[u + iv] = t[u] + t[v]$ ,
- (b) if  $u \in \text{dom } t$  is real-valued, then  $|u| \in \text{dom } t$  and  $t[|u|] \leq t[u]$ ,
- (c) there is a measurable, a.e. positive function  $\omega$  such that if  $u \in \text{dom } t$  is non-negative, then  $\min(u, \omega) \in \text{dom } t$  and  $t[\min(u, \omega)] \leq t[u]$ . Moreover, there is a form core  $\mathcal{Q}$  of  $t$  such that  $\omega^{-1}\mathcal{Q}$  is dense in  $L^2(X, \omega^{2\kappa/(\kappa-1)})$ .

The main result from [13] concerns the equivalence of an estimate on the number  $N(T - V)$  of negative eigenvalues of the operator  $T + V$ , taking multiplicities into account, and the validity of a Sobolev inequality.

**Theorem 3.2.** *Under Assumption 3.1 for some  $\kappa > 1$  the following are equivalent:*

- (i)  *$T$  satisfies a Sobolev inequality with exponent  $q = 2\kappa/(\kappa - 1)$ , that is, there is a constant  $S > 0$  such that for all  $u \in \text{dom } t$ ,*

$$t[u] \geq S \left( \int_X |u|^q dx \right)^{2/q}. \quad (3.1)$$

- (ii)  *$T$  satisfies a CLR inequality with exponent  $\kappa$ , that is, there is a constant  $L > 0$  such that for all  $0 \geq V \in L^\kappa(X)$ ,*

$$N(T + V) \leq L \int_X V_-^\kappa dx. \quad (3.2)$$

*The respective constants are bounded in terms of each other according to*

$$S^{-\kappa} \leq L \leq e^{\kappa-1} S^{-\kappa}. \quad (3.3)$$

This theorem has its origins in the Li-Yau proof of the CLR inequality [19] and we refer to [13] for further references.

We now show how to apply this theorem in order to deduce a weak form of Theorem 1.4 for *convex* domains  $\Omega$ , namely,

$$N(-\Delta - (2 \text{dist}(x, \Omega^c))^{-2} + V) \leq L_N \int_\Omega V_-^{\frac{N}{2}} dx. \quad (3.4)$$

The general case is, unfortunately, more complicated and will be dealt with in the following subsection. Obviously, the (closure of the) quadratic form

$$t[u] := \int_\Omega \left( |\nabla u|^2 - \frac{|u|^2}{4 \text{dist}(x, \Omega^c)^2} \right) dx, \quad u \in C_0^\infty(\Omega),$$

in the Hilbert space  $L^2(\Omega)$  satisfies conditions (a) and (b) above. Moreover, from the identity

$$t[v\omega] = \int_\Omega \left( |\nabla v|^2 - \frac{\Delta \text{dist}(x, \Omega^c)}{2 \text{dist}(x, \Omega^c)} |v|^2 \right) \text{dist}(x, \Omega^c) dx \quad (3.5)$$

with  $\omega := \sqrt{\text{dist}(x, \Omega^c)}$  and from the fact that  $\Delta \text{dist}(x, \Omega^c) \leq 0$  as a distribution, we easily deduce that (c) is satisfied as well. Our Hardy-Sobolev-Maz'ya inequality (1.5) and (1.4) show that (i) in Theorem 3.2 is valid, and therefore lead to (3.4).

**3.2. Proof of Theorem 1.4. The general case.** The problem with the more general inequality involving the function  $D_\Omega$  is that we do not know how to verify Assumption (c). In particular, we are not aware of a useful analogue of (3.5). We can use, however, the following remark (see the end of Section 4.1 in [13]):

*Theorem 3.2 remains valid if (c) is replaced by the following condition.*

- (d) For every a.e. positive function  $W \in L^1(X) \cap L^\infty(X)$  consider the self-adjoint, non-negative operator  $\Upsilon$  in  $L^2(X, Wdx)$  associated to the quadratic form  $t[u]$ . Then  $\exp(-\beta\Upsilon)$  is an integral operator in  $L^2(X, Wdx)$  for every  $\beta > 0$ .

We are going to prove Theorem 1.4 using (d) instead of (c). For technical reasons we have to work with regularizations defined by

$$t_\varepsilon[u] := \int_\Omega \left( |\nabla u|^2 - (1 - \varepsilon) \frac{|u|^2}{4D_\Omega^2} \right) dx, \quad u \in H_0^1(\Omega), \quad (3.6)$$

with  $\varepsilon \in (0, 1]$ . As before,  $t_\varepsilon$  satisfies (a), (b) and (i) with a constant which can be chosen independently of  $\varepsilon$ . (Namely,  $S = K_N$  from Theorem 1.1.) Hence if we can verify (d) for any  $\varepsilon \in (0, 1]$ , Theorem 3.2 yields the inequality  $N(T_\varepsilon + V) \leq L \int_\Omega V_-^{N/2} dx$  for the operators  $T_\varepsilon$  associated to  $t_\varepsilon$ . Here  $L$  is a constant independent of  $\varepsilon$ . Similarly as in [12] one can show that  $T_\varepsilon + V \searrow T_0 + V$  in strong resolvent sense. Therefore, if  $P_\varepsilon$  and  $P_0$  are the spectral projectors of  $T_\varepsilon + V$  and  $T_0 + V$  corresponding to  $(-\infty, 0)$ , then  $P_\varepsilon \rightarrow P_0$  strongly, and by Fatou's lemma for traces  $N(T_0 + V) = \text{Tr } P_0 \leq \liminf_{\varepsilon \rightarrow 0} \text{Tr } P_\varepsilon = \liminf_{\varepsilon \rightarrow 0} N(T_\varepsilon + V) \leq L \int_\Omega V_-^{N/2} dx$ , as claimed.

Thus, to complete the proof of Theorem 1.4 we need to verify that  $\exp(-\beta\Upsilon_\varepsilon)$  is an integral operator in  $L^2(\Omega, Wdx)$ . Here  $W$  is a given, a.e. positive function in  $L^1(\Omega) \cap L^\infty(\Omega)$ ,  $\beta > 0$  is a constant and  $\Upsilon_\varepsilon$  is the self-adjoint, non-negative operator in  $L^2(\Omega, Wdx)$  associated with the quadratic form  $t_\varepsilon$  from (3.6). We note that  $\Upsilon_\varepsilon u = W^{-1}(-\Delta - (1 - \varepsilon)(2D_\Omega)^{-2})u$  for  $u \in C_0^\infty(\Omega)$ . Since the coefficients of this operator are not smooth, the existence of an integral kernel is not completely standard and we include a short proof.

We claim that  $\exp(-\beta\Upsilon_\varepsilon)$  is, in fact, a Hilbert-Schmidt operator in the space  $L^2(\Omega, Wdx)$ . Via the unitary mapping  $L^2(\Omega, Wdx) \ni u \mapsto \sqrt{W}u \in L^2(\Omega, dx)$ , this is equivalent to saying that the operator  $\exp(-\beta H_\varepsilon)$  in the space  $L^2(\Omega, dx)$  is a Hilbert-Schmidt operator, where  $H_\varepsilon := W^{-\frac{1}{2}}(-\Delta - (1 - \varepsilon)(2D_\Omega)^{-2})W^{-\frac{1}{2}}$ . This, in turn, will follow if we can prove that the eigenvalues  $e_j$  of the operator  $H_\varepsilon$  satisfy a bound of the form  $e_j \geq Cj^{2/N}$ , where the constant  $C$  may depend on  $W$  and  $\varepsilon$ , but is independent of  $j$ . In Lemma 3.3 below we show that a bound of this form is true for the operator  $W^{-\frac{1}{2}}(-\Delta)W^{-\frac{1}{2}}$ , where  $-\Delta$  is the Dirichlet Laplacian on  $\Omega$ . Now Davies' Hardy inequality (1.3) implies that

$$t_\varepsilon[u] \geq \varepsilon \int_\Omega |\nabla u|^2 dx, \quad u \in H_0^1(\Omega),$$

and therefore  $H_\varepsilon \geq \varepsilon W^{-\frac{1}{2}}(-\Delta)W^{-\frac{1}{2}}$  in the sense of quadratic forms. The inequality for  $e_j$  now follows from the variational principle. This completes the proof of Theorem 1.4.  $\square$

In the previous proof we used a lower bound on the  $j$ -th eigenvalue of the operator  $W^{-\frac{1}{2}}(-\Delta)W^{-\frac{1}{2}}$  in  $L^2(\Omega, dx)$ , where  $W$  is an a.e. positive function in  $L^1(\Omega) \cap L^\infty(\Omega)$ . For later purposes we state a similar bound also in dimensions one and two.

**Lemma 3.3.** *Let  $\Omega \subset \mathbb{R}^N$  and let  $\tau > 0$  if  $N = 1, 2$  and  $\tau = 0$  if  $N \geq 3$ . Let  $(\mu_j)$  be the increasing sequence of eigenvalues (counting multiplicities) of the operator  $W^{-\frac{1}{2}}(-\Delta + \tau)W^{-\frac{1}{2}}$  in  $L^2(\Omega)$ . Then*

$$\mu_j \geq C_N \|W\|_{\frac{N}{2}}^{-1} j^{\frac{2}{N}} \quad \text{if } N \geq 3,$$

and

$$\mu_j \geq C_{N,p} \tau^{1-\frac{N}{2p}} \|W\|_p^{-1} j^{\frac{1}{p}} \quad \text{if } N = 1, 2,$$

where  $p \geq 1$  if  $N = 1$  and  $p > 1$  if  $N = 2$ .

These bounds are not new. In the following proof we shall make use of the observation that  $W^{-\frac{1}{2}}(-\Delta)W^{-\frac{1}{2}}$  is the inverse of the Birman-Schwinger operator. This allows us to derive Lemma 3.3 from classical inequalities about negative eigenvalues of Schrödinger operators.

*Proof.* We denote by  $N(\mu, W^{-1/2}(-\Delta + \tau)W^{-1/2})$  the number of eigenvalues (counting multiplicities) less than  $\mu$  of the operator  $W^{-1/2}(-\Delta + \tau)W^{-1/2}$ . We shall show that

$$N(\mu, W^{-1/2}(-\Delta)W^{-1/2}) \leq C'_N \mu^{N/2} \int_{\Omega} W^{N/2} dx \quad (3.7)$$

for  $N \geq 3$  and that

$$N(\mu, W^{-1/2}(-\Delta + \tau)W^{-1/2}) \leq C'_{N,p} \mu^p \tau^{-p+\frac{N}{2}} \int_{\Omega} W^p dx \quad (3.8)$$

for  $N = 1, 2$  and  $p$  as stated in the lemma. Obviously, these bounds are equivalent to those stated in the lemma.

To prove (3.7) for  $N \geq 3$  we note that  $N(\mu, W^{-\frac{1}{2}}(-\Delta)W^{-\frac{1}{2}})$  is equal to the number of eigenvalues greater than  $1/\mu$  of the operator  $W^{\frac{1}{2}}(-\Delta)^{-1}W^{\frac{1}{2}}$ . By the Birman-Schwinger principle this number is equal to the number of negative eigenvalues of the Schrödinger operator  $-\Delta - \mu W$ . Hence (3.7) is just a restatement of the Cwikel-Lieb-Rozenblum inequality [18, 17].

In order to prove (3.8) for  $N = 1, 2$ , we use an inequality of Lieb and Thirring [20], which states that for any non-negative operators  $A$  and  $B$  and for any  $p \geq 1$ , one has  $\text{Tr}(AB^2A)^p \leq \text{Tr} A^{2p} B^{2p}$ . For us, this implies that

$$N(\mu, W^{-\frac{1}{2}}(-\Delta + \tau)W^{-\frac{1}{2}}) \leq \mu^p \text{Tr} \left( W^{\frac{1}{2}}(-\Delta + \tau)^{-1}W^{\frac{1}{2}} \right)^p \leq \mu^p \text{Tr} W^p (-\Delta + \tau)^{-p}$$

for  $p \geq 1$ . Now we use the fact that the integral kernel of  $(-\Delta + \tau)^{-p}$ , where  $-\Delta$  is the Dirichlet Laplacian, is pointwise bounded by the same integral kernel, but now with



$-\Delta$  being the Laplacian on  $\mathbb{R}^N$ . (This is true for the integral kernel of the semi-group  $\exp(\beta\Delta)$  by the maximum principle, and follows for  $(-\Delta + \tau)^{-p}$  by integration against  $e^{-\beta\tau}\beta^{p-1}d\beta$ .) Hence, we can bound

$$\mathrm{Tr} W^p(-\Delta + \tau)^{-p} \leq \int_{\Omega} W^p dx \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} \frac{d\xi}{(\xi^2 + \tau)^p} = C_{N,p} \tau^{-p+\frac{N}{2}} \int_{\Omega} W^p dx.$$

Here the constant  $C_{N,p}$  is finite for any  $p \geq 1$  if  $N = 1$  and for any  $p > 1$  if  $N = 2$ . This proves (3.8).  $\square$

**3.3. Proof of Theorem 1.5.** In [13] it was shown that Theorem 3.2 has the following consequence.

**Corollary 3.4.** *Assume that*

- (i')  *$T$  satisfies a Sobolev interpolation inequality with  $2 < q < \infty$  and  $0 < \theta < 1$ , that is, there is a constant  $S > 0$  such that for all  $u \in \mathrm{dom} t$ ,*

$$t[u]^\theta \|u\|^{2(1-\theta)} \geq S \left( \int_X |u|^q dx \right)^{2/q}. \quad (3.9)$$

Moreover, suppose that Assumption 3.1 holds with  $\kappa$  replaced by  $q/(q-2)$ . Define  $0 < \kappa < \infty$  and  $0 < \gamma < \infty$  by

$$\gamma = \frac{q(1-\theta)}{q-2}, \quad \kappa = \frac{q\theta}{q-2}, \quad (3.10)$$

Then for all  $\tilde{\gamma} > \gamma$  and for all  $V \in L^{\tilde{\gamma}+\kappa}(X)$  the negative eigenvalues  $E_j$  of  $T + V$  satisfy

$$\sum_j |E_j|^{\tilde{\gamma}} \leq L_{\tilde{\gamma}} \int_X V_-^{\tilde{\gamma}+\kappa} dx \quad (3.11)$$

with

$$L_{\tilde{\gamma}} \leq \frac{\tilde{\gamma}^{\tilde{\gamma}+1}}{\gamma^{\gamma}(\tilde{\gamma}-\gamma)^{\tilde{\gamma}-\gamma}} \frac{\Gamma(\gamma+\kappa+1)\Gamma(\tilde{\gamma}-\gamma)}{\Gamma(\tilde{\gamma}+\kappa+1)} e^{\gamma+\kappa-1} (\theta^{-\theta}(1-\theta)^{-1+\theta} S)^{-\gamma-\kappa}.$$

Corollary 3.4 implies Theorem 1.5 in the same way in which Theorem 3.2 implies Theorem 1.4. Assumptions (a) and (b) are clearly satisfied for the quadratic form (3.6), and Theorem 1.2 gives (i') with a constant independent of  $\varepsilon$ . Moreover, since Corollary 3.4 follows from Theorem 3.2 applied to the operator  $T + \tau$  (where  $\tau > 0$  is an arbitrary parameter), Assumption (c) can be replaced by the analogue of (d) where, however,  $\Upsilon$  has to be replaced by the operator  $\Upsilon^{(\tau)}$  corresponding to the quadratic form  $t[u] + \tau\|u\|^2$ .

Similarly as in the previous subsection, we verify this condition by showing that the operator  $\exp(-\beta H_\varepsilon^{(\tau)})$  in the space  $L^2(\Omega, dx)$  is Hilbert-Schmidt with  $H_\varepsilon^{(\tau)} := W^{-\frac{1}{2}}(-\Delta - (1-\varepsilon)(2D_\Omega)^{-2} + \tau)W^{-\frac{1}{2}}$ . The latter condition is derived as before from the lower bound on the eigenvalues of the operator  $W^{-\frac{1}{2}}(-\varepsilon\Delta + \tau)W^{-\frac{1}{2}}$  stated in Lemma 3.3. This concludes the proof of Theorem 1.5.  $\square$

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